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Effective Lagrangian for low-lying states of interacting bosonic strings

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Abstract. The spectrum of a bosonic string is affected by interaction processes where the string undergoes splitting and rejoining. For low-lying states, it is possible to construct effective Lagrangians and Hamiltonians which describe the effects of the interaction, at least in the ultraviolet limit, i.e. when the splitting and rejoining process proceeds very rapidly. The derivation is performed by considering the partition function of the system consisting of an open bosonic string and integrating some of the ultraviolet degrees of freedom.

1. Statement of the problem

A well established approach to dealing with quantum systems of many degrees of freedom is the separation of those degrees which are directly involved in the particular problem under consideration from those whose presence manifests only in an indirect, although significant, way. When the degrees of freedom can be characterised by a proper frequency, this very general attitude gives rise to a separation of lower frequencies, directly involved in the problem, and higher frequencies, whose effect is due to their coupling to the lower ones. A celebrated example of this procedure may be found in the effective Lagrangian of Heisenberg and Euler, describing the photon-photon interaction mediated by charged massive particles [1, 2]. In this case the separation between higher and lower frequencies is suggested by the general features of the dynamics, because the non-existence of massless charged particles provides an intrinsic separation; it is given by the mass of the lightest charged particles.

The general procedure has, however, a wider field of application and the elimination of the higher frequencies, so that the remnant of their existence survives in an effective interaction among the lower frequencies, is an essential ingredient of the Wilson treatment and applications of the renormalisation group, both in quantum field theory and in many-body theory [3]. Here the same approach is extended to string theory. A free bosonic string is a system of uncoupled oscillators, so if we eliminate the higher frequencies no effect on the lower frequencies will be found, but as the strings interact, by splitting and rejoining, then the oscillators become coupled and the higher frequency dynamics reflects in a non-trivial way on the lower frequencies. In order to study in detail these effects a well defined system is chosen. The system is an open bosonic string, described in a light-cone gauge so that attention is paid only to the transverse dynamics [4]. In this system there is no natural boundary between high and low frequencies, since there is a unique dimensional scale, so the boundary must somehow be set up arbitrarily: the idea is to select, among the processes of splitting and rejoining

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which give rise to the string interaction [5, 6], ones that last a short time only and to look at their effects on the lower frequencies; in other words it is required that the time in which the string remains split is short with respect to the period of the oscillations which are considered. It is evident that the light-cone gauge, besides providing a very important technical simplification, making the action quadratic, also gives conceptual transparency because of the possibility of identifying the τ parameter with the time and the σ parameter with the longitudinal momentum (up to a dimensional constant), making clear the conservation of this quantity in the interaction. There are also unwelcome properties of this gauge: the covariance is lost and, for the open string case, a four-string interaction should be added [6]. The approach is to consider the present treatment as a model, in the hope that an understanding of its behaviour will allow a generalisation into a covariant treatment.

It is well known that the idea of getting an effective action, or Lagrangian, for the low-lying states of a string is by no means a new one and has received much attention for a long time. The most common approach has been to fix external low-lying states and then to evaluate the interaction induced among them, both by virtual states of the same level and also by higher states which have been pushed very far by the small slope approximation [7]; such a procedure has been recently considered also in string field theories [8]. Here the idea is different; the specification of the low-lying states will be done only at the end so, within the limitations inherent in the treatment, the result will hold for any arbitrary set of states, provided we look only at the effect of those virtual states which lie higher than the chosen set. One is really looking at the ultraviolet effect, i.e. just to the contribution of the states that are pushed far away by the small slope approximation. In this sense the treatment could be considered complementary to the more usual one; what in turn is lacking is the effect of the virtual states lying on the same level as the external ones.

The layout of the paper is as follows: after an outline of the general procedure, the construction of an interaction term, by integrating over higher frequencies is carried out. It is explicitly shown that, in the limit of fast splitting and rejoining processes, a local effective Lagrangian can be derived, starting from the general partition function. External states are not introduced at the beginning but once an effective Hamiltonian for the interaction has been derived, then its effects on the states of a free string may be studied.

2. Outline of the general procedure

The programme stated in the introduction is now put in a form which is made more and more quantitative. With a notation which is still very symbolic the task is to perform this sum:

$$\mathcal{Z} = \sum_I \exp(-\mathcal{A}_I)$$

and the sum is performed over the two-dimensional surfaces, embedded in the transverse space; the Euclidean version of the formalism will be used throughout the paper and the dynamics is formulated in first quantised version, not in string field theory. $\mathcal{Z}$ is the partition function; as already explained, external states are not considered.

The sum can be organised in a more systematic way by considering, firstly, the string propagating freely between some initial final 'time' parameters (0 and β),

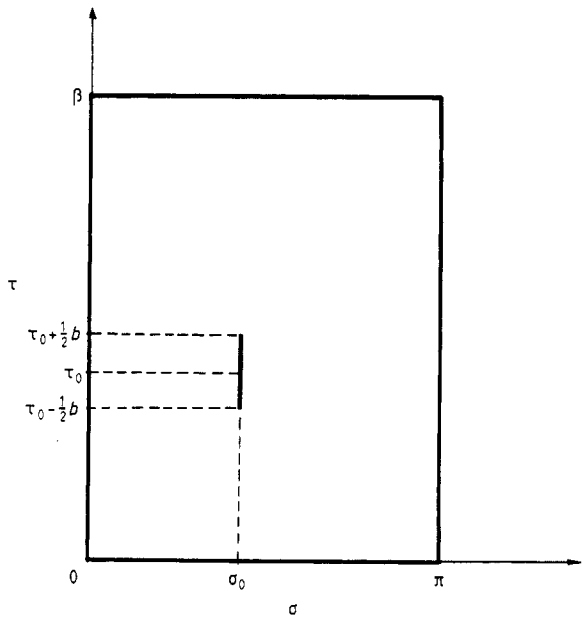

Figure 1. The parameters σ_0, τ_0 and b which define the process of breaking and rejoining of the open string.

secondly, the string which splits and rejoins once and, thirdly, the same process occurring twice and so on:

$$\mathcal{Z} = \sum_{C_0} e^{-\mathcal{A}} + \lambda \sum_{C_1} e^{-\mathcal{A}} + \lambda^2 \sum_{C_2} e^{-\mathcal{A}} + \dots . \tag{2.1}$$

The usual formulations of the interacting string theory state that the action is the same in the different addenda, but what varies is the functional space over which the sum must be performed [5]. This statement is kept also in building up string field theory, where the interaction term is also intended to describe the splitting and rejoining of the string [6]. The weights λ are, for the moment, only a sort of book-keeping introduced for future convenience.

Specifying better the content of (2.1) one can write

$$\mathcal{Z} = \int_{C_0(0,\beta)} \mathcal{D}x\, e^{-\mathcal{A}} + \lambda \int d\sigma_0\, d\tau_0 \int_{C_1(0,\beta;\sigma_0,\tau_0;b)} \mathcal{D}x\, e^{-\mathcal{A}} + \dots \tag{2.2}$$

where $C_0(0, \beta)$ is the space of functions of σ and τ ($0 \leqslant \sigma \leqslant \pi$, $0 \leqslant \tau \leqslant \beta$), which have first derivatives both with respect to σ and to τ, and have such boundary conditions that they admit an expansion in series of $\cos l\sigma$, $l = 0, 1, \dots$, so in particular $x'(0) = x'(\pi) = 0$.

C_1 is defined as follows: let $\tau' = \tau_0 - \frac{1}{2}b$, $\tau'' = \tau_0 + \frac{1}{2}b$, then for $0 \leqslant \tau \leqslant \tau'$ and $\tau'' < \tau < \beta$ the functions behave as in C_0. When $\tau' < \tau < \tau''$, the functions are, in general, discontinuous in $\sigma = \sigma_0$ and the point σ_0 acts as an endpoint, so there are two families of functions, which may be expanded respectively in series of $\cos(l\pi\sigma/\sigma_0)(0 \leqslant \sigma < \sigma_0)$ and $\cos[l\pi(\pi - \sigma)/\rho_0]$; $\rho_0 = \pi - \sigma_0$.

With these definitions, which will be explained in more detail and elaborated in what follows, a formal manipulation is performed, using the decomposition with respect

to the time parameter of the Feynman integral:

$$\int d\sigma_0\, d\tau_0 \int_{C_1} \mathcal{D}x\, e^{-\mathcal{A}}$$

$$= \int d\sigma_0\, d\tau_0 \int \mathcal{D}x(\tau')\mathcal{D}x(\tau'') \int_{C_0(0,\tau')} \mathcal{D}x\, e^{-\mathcal{A}}$$

$$\times \int_{C_1(\tau',\tau'')} \mathcal{D}x\, e^{-\mathcal{A}} \int_{C_0(\tau'',\beta)} \mathcal{D}x\, e^{-\mathcal{A}}$$

$$= \int d\sigma_0\, d\tau_0 \Bigg[\Bigg(\int \mathcal{D}x(\tau')\mathcal{D}x(\tau'') \int_{C_0(0,\tau')} \mathcal{D}x\, e^{-\mathcal{A}}$$

$$\times \int_{C_0(\tau',\tau'')} \mathcal{D}x\, e^{-\mathcal{A}} \int_{C_0(\tau'',\beta)} \mathcal{D}x\, e^{-\mathcal{A}} \Bigg) \int_{C_1(\tau',\tau'')} \mathcal{D}x\, e^{-\mathcal{A}}$$

$$\times \Bigg(\int_{C_0(\tau',\tau'')} \mathcal{D}x\, e^{-\mathcal{A}} \Bigg)^{-1} \Bigg]. \tag{2.3}$$

Now, the integral in brackets reconstructs the usual integral over C_0 which does not depend on σ_0, τ_0 and the rest, denoted by $\mathcal{F}$, is the expression we are looking for†:

$$\int d\sigma_0 \int d\tau_0 \int_{C_1} \mathcal{D}x\, e^{-\mathcal{A}} = \int_{C_0(0,\beta)} e^{-\mathcal{A}} \int d\sigma_0\, d\tau_0\, \mathcal{F}[x(\tau'), x(\tau''), \sigma_0]\mathcal{D}x. \tag{2.4}$$

For small values of the parameter b, one writes also $\mathcal{F}[x(\tau_0), \dot{x}(\tau_0), \ldots, b, \sigma_0]$.

Furthermore, as will be shown, it is possible in this limit to extract a leading local term: $\mathcal{F} \to F[x(\tau_0, \sigma_0), \dot{x}(\tau_0, \sigma_0), \ldots; b]$.

So the presence of a different functional space (C_1) has been transformed into the presence of a multiplicative factor in the integrand, which is now defined over the ordinary functional space (C_0). Since the splitting can be expressed locally we can think that the presence of two breakings will not disturb each other and the third addendum in (2.1) will produce just the square of the factor F. As the breakings are indistinguishable a factor of $\frac{1}{2}$ is introduced; taking this idea further one can foresee that n splittings will produce a term $F^n/n!$ and so one can sum the addenda (including the no-splitting) and get an overall multiplicative factor $\exp[\lambda \int (d\sigma\, d\tau\, F)]$. Writing the usual expression for the action $\mathcal{A} = \int d\sigma\, d\tau\, L$, the procedure yields, in this approximation, a sort of effective Lagrangian:

$$L_{\text{eff}} = L - \lambda F. \tag{2.5}$$

This final procedure is essentially the same as defining the grand canonical partition function of a non-interacting gas. The limit of small b is relevant because what really matters is that the overlapping in τ of two breakings is unlikely, so the first possibility is that we have to work with a diluted one-dimensional gas; it will be shown (see the appendix) that the locality in σ yields a sounder foundation for the exponentiation.

† The idea of calculating the effect of the oscillations between some given boundary conditions, taking away the effects of oscillations between different boundary conditions, is borrowed from the treatment of the Casimir effect [9] and provides an ultraviolet regularisation.

3. Determination of the interaction term

In this section the detailed calculation of the factor $\mathscr{F}$ of equation (2.4) is performed.

According to the previous discussion we have

$$\mathscr{F} = \int_{C(y)} \mathscr{D}y \exp\left(- \int\int_0^{\sigma_0} L(y', \dot{y})\, d\sigma\, d\tau \right) \int_{C(\tau)} \mathscr{D}z \exp\left(- \int\int_{\sigma_0}^{\pi} L(z', \dot{z})\, d\sigma\, d\tau \right) \times \left[\int_{C(x)} \mathscr{D}x \exp\left(- \int\int_0^{\pi} L(x', \dot{x})\, d\sigma\, d\tau \right) \right]^{-1}. \tag{3.1}$$

The functions must satisfy a given boundary condition at τ' and τ'' and at $\sigma = 0,\ \sigma_0;\ \sigma = \sigma_0,\ \pi;\ \sigma = 0,\ \pi$, respectively. Setting $\tau = \tau' + bu$ the functions allow the following representations:

$$\begin{aligned} x &= x_i(1-u) + x_f u + \tilde{x} \qquad 0 \leqslant \sigma \leqslant \pi \\ y &= y_i(1-u) + y_f u + \tilde{y} \qquad 0 \leqslant \sigma \leqslant \sigma_0 \\ z &= z_i(1-u) + z_f u + \tilde{z} \qquad \sigma_0 \leqslant \sigma \leqslant \pi \end{aligned} \tag{3.2}$$

where $x_i = x(\tau')$, $x_f = (\tau'')$ and so on. The 'tilde' functions are zero at τ' and τ'' ($u = 0, 1$). x_i and x_f are assumed to be given and they enter in the integral of (2.4). y_a and z_a must coincide, within their domain of definition, with $x_a (a = \mathrm{i}, \mathrm{f})$ except for a boundary condition. Explicitly, this condition can be implemented as follows:

$$y_a = \eta_0 + \sum_l \eta_l \cos(l\pi\sigma/\sigma_0) \tag{3.3a}$$

$$\eta_0 = \frac{1}{\sigma_0} \int_0^{\sigma_0} x_a(\sigma)\, d\sigma \qquad \eta_l = \frac{2}{\sigma_0} \int_0^{\sigma_0} x_a(\sigma) \cos(l\pi\sigma/\sigma_0)\, d\sigma. \tag{3.3b}$$

Since x_a' exists in general, the representations give $y_a' = x_a'$, but at $\sigma = \sigma_0$ we have that $y_a' = 0$, while $x_a' \neq 0$. This fact reflects on the second derivative of y_a which has a δ-like singularity in σ_0 (strictly speaking, only the left derivative exists). The coefficient of this δ singularity is $-x'(\sigma_0)$ because this is the amount of the discontinuity of the first derivative of y. Sometimes the following notation will be used:

$$\begin{aligned} y_a''(\sigma_0) &= \lim_{\sigma \to \sigma_0} y_a''(\sigma) \\ \partial_\sigma^2 y_a(\sigma) &= y_a''(\sigma) - x_a'(\sigma_0)\delta(\sigma - \sigma_0) \qquad \sigma \leqslant \sigma_0. \end{aligned} \tag{3.4}$$

With obvious changes the same can be said about the function z_a, which will exhibit a δ-like singularity in its right second derivative at σ_0:

$$\begin{aligned} z_a''(\sigma_0) &= \lim_{\sigma \to \sigma_0} z_a''(\sigma) \\ \partial_\sigma^2 z_a(\sigma) &= z_a''(\sigma) + x_a'(\sigma_0)\delta(\sigma - \sigma_0) \qquad \sigma \geqslant \sigma_0. \end{aligned} \tag{3.5}$$

In order to perform the functional integration, a basis on which to expand the oscillating variables is required:

$$\begin{aligned} \tilde{x} &= \sum_n X_n \varphi_n(u) \\ \tilde{y} &= \sum_n Y_n \varphi_n(u) \\ \tilde{z} &= \sum_n Z_n \varphi_n(u) \end{aligned} \tag{3.6}$$

with $\varphi_n(0) = \varphi_n(1) = 0$.

The normalisation of the basis affects the result of the functional integral which depends therefore on an overall normalisation condition. A very reasonable requirement is $\mathcal{F} \to 1$ for $b \to 0$. This requirement is satisfied if one takes the normalisation condition

$$\int_0^1 \partial_u \varphi_m \, \partial_u \varphi_n \, \mathrm{d}u = b\delta_{m,n} \tag{3.7}$$

which gives

$$\varphi_n(u) = \sqrt{2b}(n\pi)^{-1} \sin \pi n u.$$

Now we are in a position to calculate, using equations (3.2), (3.6) and (3.7),

$$\begin{aligned}\mathcal{X} &= \tfrac{1}{2}\int_{\tau'}^{\tau''} (\dot{x}^2 + x'^2)\, \mathrm{d}\tau \\ &= \tfrac{1}{2}\Bigg\{ \sum_{n=1}^{\infty} \left[\left(X_n^2 + \frac{b^2}{n^2\pi^2} X_n'^2 \right) + X_n' \frac{\sqrt{2}b^{3/2}}{n^2\pi^2} (x_\mathrm{i}' - (-1)^n x_\mathrm{f}') \right] \\ &\quad + \tfrac{1}{3}b(x_\mathrm{i}'^2 + x_\mathrm{i}' x_\mathrm{f}' + x_\mathrm{f}'^2) + (1/b)(x_\mathrm{f} - x_\mathrm{i})^2 \Bigg\}. \end{aligned} \tag{3.8}$$

Using the condition that X_n' and x_a' must be zero at $\sigma = 0, \pi$ some partial integration can be performed and the following expression is obtained:

$$\int_0^{\pi} \mathrm{d}\sigma \mathcal{X} = \tfrac{1}{2}\int_0^{\pi} \mathrm{d}\sigma \left(\sum_n (X_n K_n^{[x]} X_n - 2X_n \partial_\sigma^2 J_n^{[x]}) + S^{[x]} \right) \tag{3.9}$$

where we have defined the operator

$$K_n^{[x]} = 1 - (b/\pi n)^2 \partial_\sigma^2 \tag{3.10a}$$

the external source

$$J_n^{[x]} = \frac{b^{3/2}}{\sqrt{2}\pi^2 n^2} [x_\mathrm{i} - (-1)^n x_\mathrm{f}] \tag{3.10b}$$

and the term independent of X:

$$S^{[x]} = \tfrac{1}{3}b(x_\mathrm{i}'^2 + x_\mathrm{i}' x_\mathrm{f}' + x_\mathrm{f}'^2) + (x_\mathrm{f} - x_\mathrm{i})^2 / b. \tag{3.10c}$$

In exactly the same way one can calculate the corresponding expressions for

$$\mathcal{Y} = \tfrac{1}{2}\int_{\tau'}^{\tau''} (\dot{y}^2 + y'^2)\, \mathrm{d}\tau \qquad \mathcal{Z} = \tfrac{1}{2}\int_{\tau'}^{\tau''} (\dot{z}^2 + z'^2)\, \mathrm{d}\tau$$

the operators K_m appearing in $\mathcal{Y}$ and $\mathcal{Z}$ have the same form as in (3.10a) but for the substitution of π by σ_0 and ρ_0, respectively; the terms J and S are modified in an obvious way. The terms S are independent of X, Y and Z, so they do not enter in the functional integration; they just give rise to constant factors which in the definition of $\mathcal{F}$ appear, on the whole, as

$$\Phi = \exp\left[\frac{1}{2} \left(\int_0^{\pi} S^{[x]}\, \mathrm{d}\sigma - \int_0^{\sigma_0} S^{[y]}\, \mathrm{d}\sigma - \int_{\sigma_0}^{\pi} S^{[z]}\, \mathrm{d}\sigma \right) \right].$$

By construction and definition of y_a, z_a it turns out that $S^{[y]}$ and $S^{[z]}$ coincide with $S^{[x]}$ in their domain of definition in σ. There can be a difference at the point σ_0, but

since only first derivatives are involved there is no room for δ singularities, the difference in the exponents gives zero and thus $\Phi = 1$; so in practice one can simply drop the term $S^{[x]}$ in (3.9) provided one does the same in the corresponding expressions for y and z.

The next step is to write down the Green function of the operator K_n. Owing to the boundary condition implied in the expansion (3.6) and (3.7) the role of the δ function is played by the function

$$D^{[x]}(\sigma, \sigma') = \frac{1}{\pi} + \frac{2}{\pi} \sum_l \cos l\sigma \cos l\sigma' \tag{3.11}$$

so that the Green function, defined by $K_n^{[x]} G_n^{[x]} = D^{[x]}$ is

$$G_n^{[x]}(\sigma, \sigma') = \frac{1}{\pi} + \frac{2}{\pi} \sum_l \frac{1}{1 + (lb/\pi n)^2} \cos l\sigma \cos l\sigma' \tag{3.12a}$$

and in a strictly analogous way we have that

$$G_n^{[y]}(\sigma, \sigma') = \frac{1}{\sigma_0} + \frac{2}{\sigma_0} \sum_l \frac{1}{1 + (lb/\sigma_0 n)^2} \cos(l\pi\sigma/\sigma_0) \cos(l\pi\sigma'/\sigma_0) \tag{3.12b}$$

$$G_n^{[z]}(\sigma, \sigma') = \frac{1}{\rho_0} + \frac{2}{\rho_0} \sum_l \frac{1}{1 + (lb/\rho_0 n)^2} \cos[l\pi(\pi - \sigma)/\rho_0] \cos[l\pi(\pi - \sigma')/\rho_0]. \tag{3.12c}$$

The Green function can be expanded in powers of b^2, for instance from equations (3.12a) and (3.11) one gets

$$G_n^{[x]} = D^{[x]} + (b/\pi n)^2 \partial_\sigma^2 D^{[x]} + \dots . \tag{3.13}$$

However, in some cases the closed form of G will be explicitly needed.

We are now ready to calculate the functional integrals appearing in the definition of $\mathscr{F}$ with the result:

$$\begin{aligned} \mathscr{F} = \prod_n & \int \mathscr{D}Y_n \exp\left(- \int_0^{\sigma_0} \mathscr{Y} \, d\sigma \right) \int \mathscr{D}Z_n \exp\left(- \int_{\sigma_0}^{\pi} \mathscr{Z} \, d\sigma \right) \\ & \times \left[\int \mathscr{D}X_n \exp\left(- \int_0^{\pi} \mathscr{X} \, d\sigma \right) \right]^{-1} \\ & = \prod_n (\det K_n^{[x]} / \det K_n^{[y]} \det K_n^{[z]})^{1/2} \\ & \times \exp\left[\frac{1}{2} \sum_n \left(\int d\sigma \, \partial_\sigma^2 J_n^{[y]} G_n^{[y]} \partial_\sigma^2 J_n^{[y]} + \int d\sigma \, \partial_\sigma^2 J_n^{[z]} G_n^{[z]} \partial_\sigma^2 J_n^{[z]} \right. \right. \\ & \left. \left. - \int d\sigma \, \partial_\sigma^2 J_n^{[x]} G_n^{[x]} \partial_\sigma^2 J_n^{[x]} \right) \right]. \end{aligned} \tag{3.14}$$

The determinant factor is calculated as usual:

$$\begin{aligned} & \Delta = \exp\left(\tfrac{1}{2} \operatorname{Tr} \sum_n (\ln K_n^{[x]} - \ln K_n^{[y]} - \ln K_n^{[z]}) \right) \\ & \ln K_n = -(b/\alpha n)^2 \partial_\sigma^2 - \tfrac{1}{2}(b/\alpha n)^4 \partial_\sigma^4 - \dots \end{aligned} \tag{3.15}$$

where $\alpha = \pi, \sigma_0, \rho_0$, respectively, for $[x], [y], [z]$. The sum over n can be performed and gives

$$\sum_n \ln K_n = -\tfrac{1}{6}(b\pi/\alpha)^2 \partial_\sigma^2 - \tfrac{1}{180}(b\pi/\alpha)^4 \partial_\sigma^4 - \dots . \tag{3.16}$$

At first sight we find a trivial divergence in $\mathrm{Tr}\,\partial_\sigma^2$, but we can apply a common regularisation, remembering that for the three cases (x, y, z) the wavenumber is respectively proportional to l/π, l/σ_0, l/ρ_0. So a common regularising procedure can be set up:

$$\sum_l l^{2n} \to \sum_l l^{2n} \exp(-\mu l/\alpha) = (\alpha\partial/\partial_\mu)^{2n}(\tfrac{1}{2}\tanh \mu/2\alpha - \tfrac{1}{2}) \tag{3.17a}$$

and the regularising parameter must go finally to zero.

Since $n > 0$, the constant term $-\frac{1}{2}$ can be dropped, the remaining term is odd in μ and the following expansion, therefore, holds [10]:

$$\tfrac{1}{2}\tanh \mu/2\alpha = \frac{\alpha}{\mu} + \sum_{m=0}^{\infty} c_{2m+1}\left(\frac{\mu}{\alpha}\right)^{2m+1}. \tag{3.17b}$$

Performing the subtraction implied in (3.15) in the limit $\mu \to 0$, only the singular term survives because we always take an even number of derivatives (see equation (3.17*a*)), while the powers of μ are always odd (see equation (3.17*b*)). So we get in the limit, for the term ∂_σ^2,

$$-\frac{b^2}{3\mu^3}\left[\pi^3 - \left(\frac{\pi}{\sigma_0}\right)^2 \sigma_0^3 - \left(\frac{\pi}{\rho_0}\right)^2 \rho_0^3\right] = 0. \tag{3.18}$$

The same happens for the higher terms because the diverging parts compensate while the finite parts go to zero; in conclusion it results that†

$$\Delta = 1.$$

In calculating the part containing the source terms J explicit use is made of the idea that b is small, and so, whenever possible, the expansion of equation (3.13) will be used. When considering the x term there are no problems with the second derivatives $\partial_\sigma^2 J = J''$ and we can write a factor

$$\exp\left(-\frac{1}{2}\sum_n \int [J_n'' J_n'' + (b/n\pi)^2 J_n'' J_n'^{v} + \ldots]\,\mathrm{d}\sigma\right). \tag{3.19}$$

According to the definition (3.10*b*), the first term in the exponent is of order b^3 and the second is of order b^5.

For the y term we must take into account explicitly the property of $\partial_\sigma^2 J^{[y]}$ as in (3.4) so that the following expression is produced:

$$\exp\left[\frac{1}{2}\sum_n \left(\int [J_n^{[y]''} G_n^{[y]} J_n^{[y]''} - J_n^{[x]'}(\sigma_0) G_n^{[y]}(\sigma_0, \sigma) J_n^{[y]''} - J_n^{[y]''} G_n^{[y]}(\sigma, \sigma_0) J_n^{[x]'}]\,\mathrm{d}\sigma + J_n^{[x]'}(\sigma_0) G_n^{[y]}(\sigma_0, \sigma_0) J_n^{[x]'}(\sigma_0)\right)\right]. \tag{3.20}$$

There is an analogous expression with some change of signs for the z term.

Collecting all the terms in b^3, using the expansion of the Green function we get

$$\tfrac{1}{2}b^3 \sum_n \left[\left(\int_0^{\sigma_0} J_n^{[y]''} J_n^{[y]''}\,\mathrm{d}\sigma + \int_{\sigma_0}^{\pi} J_n^{[z]''} J_n^{[z]''}\,\mathrm{d}\sigma - \int_0^{\pi} J_n^{[x]''} J_n^{[x]''}\,\mathrm{d}\sigma\right) + [2J_n^{[x]'}(\sigma_0) J_n^{[z]''}(\sigma_0) - 2J_n^{[x]'}(\sigma_0) J_n^{[y]''}(\sigma_0)]\right]$$

† See the appendix for further consideration on the method of calculating Δ.

but $J^{[y]''}$ (and $J^{[z]''}$) coincide with $J^{[x]''}$ in their domains of definition and therefore the whole coefficient of the term in b^3 is zero; the next contribution, of order b^5, will not, in general, compensate.

We are now left with the terms that are local in σ_0. In this case the contribution from the y and the z terms simply add together; out of expression (3.12*b*, *c*) we get [10]

$$G_n^{[y]}(\sigma_0, \sigma_0) = (\pi n/b)\,\mathrm{cotanh}(\pi n\sigma_0/b)$$

$$G_n^{[z]}(\sigma_0, \sigma_0) = (\pi n/b)\,\mathrm{cotanh}(\pi n\rho_0/b).$$

For $b \to 0$ the two expressions merge into

$$G_n^{[y]} \approx G_n^{(z)} \approx \pi n/b$$

to the exponentially vanishing terms so, for this particular contribution, which is leading for $b \to 0$, it results that $T_2 = \Sigma_n\,(\pi n/b)(J_n^{[x]'}(\sigma_0))^2$, which is of order b^2.

Using the expression of the sources $J^{[x]}$ and performing the sum† over n

$$T_2(\sigma_0) = \tfrac{1}{2}(2\pi)^{-3}\zeta(3)b^2[7(x_i' + x_f')^2 + (x_i' - x_f')^2]/\sigma_0. \tag{3.21}$$

This expression represents the result we were looking for, and we can in fact write:

$$\mathscr{F} = \mathrm{e}^{T_2}.$$

Remembering that, in getting the expression for T_2 the terms discarded were exponentially small (of the type $(\mathrm{e}^{-c/b})/b$) and that the terms in b^3 compensate to zero, we conclude that the first correction to T_2 should be of order b^5 and probably quite complicated and, in particular, non-local in σ_0.

4. Effective Lagrangian and Hamiltonian

Within the scheme of considering the limit $b \to 0$ it is useful to elaborate further the expression of T_2. With the symbol $x_c = x(\tau_0)$ we write

$$[7(x_i' + x_f')^2 + (x_i' - x_f')^2] \to 28x_c'^2 + 7b^2 x_c' \ddot{x}_c' + b^2 \dot{x}_c'^2$$

so that we keep in T_2 terms of order b^2 and b^4.

Now we have for the interaction term an expression local both in σ and in τ and we can, therefore, proceed to the exponentiation, in order to take into account the possibility of many 'non-interacting' splittings and we get, therefore, a term

$$\exp\left(\lambda \int \mathrm{d}\sigma_0\,\mathrm{d}\tau_0 \exp(T_2(\sigma_0, \tau_0))\right). \tag{4.1}$$

It is anyhow consistent with the approximations to expand the exponential inside the integral, to order b^4. In so doing a partial integration in τ (with the elimination of the total derivative) unifies the term in $\dot{x}'^2$ with the term in $x'\ddot{x}'$.

The overall result can be expressed by introducing an effective Lagrangian

$$L_{\mathrm{eff}} = L_0 - \lambda L_{\mathrm{int}}$$

$$L_{\mathrm{int}} = 1 + \frac{7}{4\pi^3}\zeta(3)b^2 x'^2 - \frac{3}{8\pi^3}\zeta(3)b^4 \dot{x}'^2 + \frac{1}{2}\left(\frac{7}{4\pi^3}\zeta(3)\right)^2 b^4 x'^4. \tag{4.2}$$

† With standard notation [10] $\zeta(3) = \Sigma\, k^{-3} = 1.202\ldots$.

Up until now we have worked with the Feynman path integral formalism and in Euclidean formulation.

Since in L_{eff} we have only quadratic terms in $\dot{x}$ and $\dot{x}'$, if we perform back time rotation and simultaneously define the Hamiltonian in the usual way, the expression of equation (3.1) yields the Hamiltonian density, so that starting from (4.2) we have an effective interaction Hamiltonian

$$\mathscr{H}_{\text{int}} = -\lambda \int L_{\text{int}} \, \mathrm{d}\sigma. \tag{4.3}$$

It is also possible to set up the standard quantisation procedure and study the effect of these new terms on the spectrum.

First of all there is a finite numerical term $-\lambda\pi$, which shifts the zero-point energy. The other terms give rise, together with the unperturbed Hamiltonian, to the following expression†:

$$\mathscr{H}_{\text{tot}} = \tfrac{1}{2} \int [(1+c_1)x'^2 + \dot{x}^2 - c_2\dot{x}'^2 + c_3 x'^4] \, \mathrm{d}\sigma. \tag{4.4}$$

The first terms can be diagonalised exactly leading to the spectrum

$$\mathscr{H}_{(2)} = \sum_n [(1+c_1)/(1-c_2 n^2)]^{1/2} n a_n^\dagger a_n.$$

We see that the term c_1 (which is of order b^2) simply changes the spacing of the spectrum, the string tension, while the term c_2 (which is of order b^4) partially breaks the degeneration of the spectrum: in fact, a state $a_1^\dagger a_1^\dagger|\rangle$ is no longer degenerate with $a_2^\dagger|\rangle$. It is explicitly seen that the expression for $\mathscr{H}_{(2)}$ is inconsistent for a too high value of n; this reflects a very general fact that, if the frequencies which have been integrated over (which are of the order $1/b$ or higher) are not much higher than the frequencies of the physical states which we look at, we shall find a lack of unitarity which shows up as a non-Hermitian expression in the Hamiltonian. Taking into account those limitations it would be more realistic to write

$$\mathscr{H}_{(2)} = \sum_n [(1+c_1)^{1/2} + \tfrac{1}{2}c_2 n^2] n a_n^\dagger a_n. \tag{4.5}$$

The effect of the third addendum, the anharmonic term, is, at first order, to produce a further breaking of the degeneracies of the unperturbed spectrum. Up until now the vectorial properties of transverse degrees of freedom have never been taken into account explicitly, for notational simplicity, but more precisely it should be written, denoting by d the number of transverse dimensions,

$$\mathscr{H}_{(3)} = \tfrac{1}{2}c_3 \int (x_k x_k)^2 \, \mathrm{d}\sigma \qquad k = 1, \ldots, d. \tag{4.6}$$

Being a perturbation, its action can be studied on the unperturbed states and, since it involves the rotation degrees of freedom in d dimensions, a general treatment is complicated, even though straightforward in principle. A particular example, therefore, will be worked out. Two states of the type $a_1^\dagger a_1^\dagger|\rangle$ will be considered, a trace state $|t\rangle = (a_1^\dagger)_k (a_1^\dagger)_k|\rangle$ and a traceless state $|s\rangle = \eta_{jk}(a_1^\dagger)_j (a_1^\dagger)_k|\rangle$, with $\eta_{jj} = 0$, both normalised, of course. The perturbation $\mathscr{H}_{(3)}$ cannot connect these two orthogonal states and the

† The values of c_i can be read off from (4.2); they are all linear in λ.

relevant matrix elements are therefore the diagonal ones. We start by looking at the part of $\mathscr{H}_{(3)}$ which is built up with a_1 operators: moreover only the part containing two creation and two destruction operators is relevant. In conclusion, we extract out of the perturbation the term

$$\mathscr{H}_{(3,\mathrm{eff})} = \frac{3}{16\pi} c_3[(2a_k^\dagger a_k + d + 1)^2 - 1 + 2a_k^\dagger a_k^\dagger a_j a_j].$$

The mode index '1' is understood. Now while the first two addenda give the same matrix element in $|t\rangle$, we have for the third term that

$$\langle t|a_k^\dagger a_k^\dagger a_j a_j|t\rangle = 2d$$
$$\langle s|a_k^\dagger a_k^\dagger a_j a_j|s\rangle = 0$$

so the splitting between the states $|t\rangle$ and $|s\rangle$ induced by $\mathscr{H}_{(3)}$ is

$$\Delta E_3 = (3/4\pi) c_3 d.$$

A more complicated splitting pattern can be produced looking at higher states†.

Besides these shifts, state mixing is also produced. For instance, there is a mixing between the $|t\rangle$ state and the vacuum state; it is clear that more generally the vacuum state can get contributions from the two-particle and four-particle scalar states, already at first order.

In dealing with the effective Hamiltonian the parameter λ acts as a coupling constant, but the time parameter b plays an analogous role. Until now b has been taken as a constant, but in fact it could also be allowed to vary, below a fixed limit. In this case the expression (2.2) could be replaced by

$$\mathbb{Z} = \int_{c_0} \mathscr{D}x\, \mathrm{e}^{-\mathscr{A}} + \lambda \int_0^{b_M} \mathrm{d}b \int \mathrm{d}\sigma_0\, \mathrm{d}\tau_0 \int_{c_1} \mathscr{D}x\, \mathrm{e}^{-\mathscr{A}} + \ldots .$$

Following throughout the derivation and the changes induced in this way up until equations (4.2) and (4.4), we see that trivial modifications occur. The renormalisation term becomes $-\lambda b_M \pi$ and the substitutions $b^2 \to \frac{1}{3} b_M^3$, $b^4 \to \frac{1}{5} b_M^5$, are required; the first neglected term is now of order b_M^6.

At first sight it appears that b acts as a coupling constant but looking at its original meaning we can call it, and especially b_M, a sort of cutoff. It has in this treatment the role of an infrared cutoff because the effect of interactions lasting longer than b_M is neglected. The parameter λ could also be described as a coupling constant; its main role is, however, to be a normalisation factor, related to the integrations over the parameters describing the splitting and rejoining of the string, both numerical (σ_0, τ_0, b) and functional (X, Y, Z). No difficulty arises setting $\lambda = 1$, but one must remember that this choice is just a particular normalisation of the integrals, not the elimination of a parameter which appears to be free, but for the limitation of being real and positive.

5. Conclusions

The main result of this investigation is the construction of an effective action and then an effective Hamiltonian, which represents, in the functional space of the free string and in a well defined limit, the effects of the splitting. The effective Hamiltonian gives rise both to a distortion in the spectrum and to a coupling among the modes; in this

† Also in states like $|t\rangle$ and $|s\rangle$, operational structures like $a_1^+ \cdot a_2 a_2^+ \cdot a_1$ give non-zero contributions.

respect the result is qualitatively different from what happens in the prototype Euler-Heisenberg Lagrangian: for an interacting string the dynamical system is more complicated, having infinitely many degrees of freedom, but it is also unique in the sense that the loops have the same constituents as the possible external states. It happens that the interaction among the system as a whole, i.e. the strings, produces an interaction inside the system, the coupling of the modes, besides the distortion of the mode spectrum. With the definition assumed in performing the functional integrals, no ultraviolet divergence has been found. The results stays quite finite in the limit $b \to 0$, although it is not analytical for $b = 0$, because exponentially vanishing terms are produced. The reason for this absence of ultraviolet divergences is found in the fact that the ultraviolet excitations for the interacting case tend to become more and more equal to the free case excitations as the frequency grows. Within the model some problems still remain open, in particular the question of the possibility of exponentiation if further terms, non-local in σ, are taken into account. A more conceptual point concerns the role of the parameter λ, which is strictly tied to the normalisation of the integrals used in defining the interaction term $\mathcal{F}$. It must also be noted that only planar configurations of the evolution surface, that go into the free-string surface, when the duration of the splitting goes to zero, have been considered, disregarding all more complicated topologies that can be produced in the general evolution.

A question that also arises is whether some renormalisation is required. The need for renormalisation is independent of the presence of infinities, and it could be required even if only finite quantities appear. There are two conditions which appear natural; each of them can be satisfied but not both together. The first is that there is no shift for the zero-point energy, i.e. for the intercept we still have $\alpha_0 = 1$. This condition is equivalent to asking that for $b \to 0$, $\mathcal{F} \to 0$, while in the formalism presented in this paper $\mathcal{F} \to 1$. The other possibility is that the first-level states, the 'photons', remain massless; this request can be implemented by another, b-dependent, renormalisation of the intercept.

This second kind of renormalisation condition is suggested by the attempt of comparing the procedure presented with other kinds of effective actions which have been proposed. In particular, a very detailed treatment is available for the effective action of the first levels of the open string [11], but not only the treatment but also the starting point are very different. In the references quoted the level one is singled out of the rest and treated as a massless field, the other degrees of freedom are integrated over, and so the outcome is an effective interaction among massless particles. Higher states are not dealt with explicitly so that the question of the relative shifts or splitting of the levels cannot be asked. On the other hand the result is exact (there is nothing like the b parameter), it is covariant and can be used to describe the scattering of massless particles.

The treatment shown in the present paper applies to static properties of the spectrum, and not to scattering processes, since the basic tool is the partition function. It also refers to higher levels, but it is never complete since it depends on b. In conclusion, the treatment appears more complementary than alternative and there is no region where a quantitative comparison can be set up.

The really relevant task is, however, the extension of this kind of treatment, where no special role is played by the massless levels, to a covariant formulation, perhaps in the form made fashionable by Polyakov [11] in order to have a quadratic action to start with. Here a major problem is expected to arise from the redundancy of the variables which are not in one-to-one correspondence with the physical degrees of

freedom [4, 12]. Of course, parallel to this extension, the inclusion of fermionic degrees of freedom could also be attempted.

Appendix

In this appendix some further arguments concerning points previously treated are presented. The first concerns the exponentiation of the interaction; it is in fact possible to complete the argument for situations that have not yet been taken into account. Let us consider a particularly bad situation, where at a time τ_0 two splitting processes take place, at the points σ_1 and σ_2.

Then we can consider (in comparison with equation (3.2)) a representation

$$\begin{aligned} x &= x_i(1-u)+x_f u+\tilde{x} && 0\leqslant\sigma\leqslant\pi \\ y &= y_i(1-u)+y_f u+\tilde{y} && 0\leqslant\sigma\leqslant\sigma_1 \\ w &= w_i(1-u)+w_f u+\tilde{w} && \sigma_1\leqslant\sigma\leqslant\sigma_2 \\ z &= z_i(1-u)+z_f u+\tilde{z} && \sigma_2\leqslant\sigma\leqslant\pi. \end{aligned}$$

The properties of y_a and z_a are the same as before, while it is clear that for w it results that

$$\partial_\sigma^2 w_a(\sigma) = w_a''(\sigma)+x_a'(\sigma_1)\delta(\sigma-\sigma_1)-x_2'(\sigma_2)\delta(\sigma-\sigma_2).$$

Then y, w and z are expanded on the basis given in (3.7) and all the procedure is as before; in particular, the terms of type S compensate. In calculating the determinant an evident adaptation of the regularisation procedure yields $\Delta = 1$. The other compensations, after integration over X, Y, W and Z, also hold and the remaining contributions of the singularities of the second derivatives finally give rise to a term

$$T_2(\sigma_1,\sigma_2) = T_2(\sigma_1)+T_2(\sigma_2)$$

where $T_2(\sigma_1)$ is defined as in (3.21).

So in conclusion we get

$$\mathscr{F} = \mathrm{e}^{T(\sigma_1)}\,\mathrm{e}^{T(\sigma_2)}$$

which is the factorisation required to allow the exponentiation as in equation (4.1).

The second point concerns the regularisation in calculating the functional determinant Δ. One can ask how much the result depends on the particular regularising procedure. Here a general argument is not presented, but the particular regularisation previously used is changed, taking instead of equation (3.17),

$$\begin{aligned} \sum_l l^{2n} &\to \sum_l \frac{1}{n!}\left(-\frac{\partial}{\partial\kappa^2}\right)^n \frac{1}{1+(l\kappa)^2} \\ &= \frac{1}{n!}\left(-\frac{\partial}{\partial\kappa^2}\right)^n \left(\frac{\pi}{2\kappa}\operatorname{cotanh}\frac{\pi}{\kappa}-\frac{1}{2}\right) = f(\kappa) \end{aligned}$$

where $\kappa = \mu/\alpha$. Now taking the limit $\mu \to 0$, we have that

$$\begin{aligned} f(\kappa) &= \frac{1}{n!}\left(-\frac{\partial}{\partial\kappa^2}\right)^n \left(\frac{\pi}{\kappa}(1+2\exp(-2\pi/\kappa)+\ldots)-\tfrac{1}{2}\right) \\ &= \frac{\pi}{2}\frac{(2n-1)!!}{(2n)!!}\left(\frac{\alpha}{\mu}\right)^{2n+1} + \mathrm{O}(\exp(-C/\mu)). \end{aligned} \tag{A1}$$

Looking at (3.16), we see that in any case only a term of the kind $(\pi - \sigma_0 - \rho_0)$ is obtained and so the compensation that finally yields $\Delta = 1$ holds to terms exponentially vanishing in the regularising parameter μ. The same result is obtained through a Gaussian regularisation,

$$\sum_l l^{2n} \to \sum_l (-\partial/\partial\kappa)^n \exp(-\kappa l^2) = \tfrac{1}{2}(-\partial/\partial\kappa)^n \theta_3(0, e^{-\kappa}) = g(\kappa).$$

Applying now the Jacobi imaginary transformation [13] to the function θ_3, we obtain

$$g(\kappa) = \tfrac{1}{2}(-\partial/\partial\kappa)^n (\pi/\kappa)^{1/2} \theta_3(0, \exp(-\pi^2/\kappa)).$$

In this case the parameter κ must be identified with μ^2/α^2 and, expanding the θ_3, we easily reach the result that a formula like (A1) holds and so the same final conclusions may be drawn.

So the really significant fact is that, when the cutoff is introduced, the number of modes is proportional to α. This is relevant for a related problem: the actual calculation of the determinant appears dependent on a normalisation of the basis functions, which was explicitly fixed in (3.7). A change in the normalisation such as $\varphi_n \to \hat{\varphi}_n = C_n(b)\varphi_n$ induces a change in the functional integration variables $X_n(\sigma) \to X_n(\sigma)/C_n(b)$ and so on. In turn, if the functional variable has α/μ modes we get a factor $[C_n(b)]^{-\alpha/\mu}$ so that the whole determinant (3.15) is changed, remembering the values taken by α, by

$$[C_n(b)]^{(-\pi+\sigma_0+\rho_0)/\mu}$$

so, taking correctly into account the regularisation of the l modes, the result $\Delta = 1$ is obtained independently of the basis.

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